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Interfacial Approach to d-Dimensional $\pm J$ Ising Models in the Neighborhood of the Ferromagnetic Phase Boundary

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Two- and three-dimensional $\pm J$ Ising models in the neighborhood of the ferromagnetic phase (FP) boundary in the concentration–temperature (p–T) plane are studied, investigating the size dependence of interfacial free energies calculated by a transfer matrix method. The p and T dependences of two stiffness exponents relevant to the FP and the nonferromagnetic ordered phase lead to the following results in two dimensions, giving a unified view. It is confirmed that the random antiphase state (RAS) exists in contact with the vertical FP boundary. Spatial fluctuations are dominant near the vertical boundary, which is separated by the Nishimori line from the remaining FP boundary governed by thermal fluctuations. The RAS is a kind of Mattis spin glass such that it changes to the FP smoothly with nonsingular physical connectivity, but with a percolation singularity of its ferromagnetic part. Universal finite-size critical amplitudes are consistent with them. Results in three dimensions give only suggestions which are similar to the two-dimensional results. These results suggest important insight into spin-glass properties in higher dimensions.

KEY WORDS: $\pm J$ model; interfacial free energy; transfer matrix method; stiffness exponent; random antiphase state; Nishimori line; Mattis spin glass; spatial fluctuation; thermal fluctuation.

1. INTRODUCTION

In the last decade interest and effort in the spin-glass (SG) problem have focused mainly on short-range SG models, especially on the $\pm J$ Ising model.[1] On one hand, for the symmetric distribution of the $\pm J$ model in three dimensions (3D), large scale Monte Carlo simulations revealed the existence of an SG phase transition at a nonzero temperature,[2,3] in

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contrast to its absence in 2D.[4] These results are consistent with those obtained by scaling analyses of interfacial free energies[5,6] and were confirmed by high-temperature expansion.[7] On the other hand, for asymmetric distributions of general nearest-neighbor Ising models, some interesting properties have been found when no external field is applied. One is the so-called Nishimori line (N-line) in the concentration–temperature (p–T) plane along which there is no singularity in energy.[8] Second is the vertical phase boundary along the T axis of the ferromagnetic phase (FP).[9] There are theoretical[10–12] and experimental studies consistent with it; see ref. 11 for experimental ones. Third is the random antiphase state (RAS) which exists adjacent to the FP.[13,14] Ground-state properties of the RAS in 2D have been studied intensively,[13,14] but thermodynamic properties in 2D and ground-state properties in higher dimensions are not so clear.

For convenience below we shall generically call the ordered phase without uniform magnetization the random phase (RP), which can be the RAS, SG, or other nonferromagnetic ordered phases, confining the definition of SG phase to the one found at the symmetric distribution. In this sense the latter three properties are not directly associated to SG phases, but there are many interesting questions about them or generally the behavior in the neighborhood of the FP boundary:

1. Does the FP have common properties in all the FP region?
2. What does the N-line really indicate?
3. Why does the FP have a vertical phase boundary if it is true?
4. Does the RAS really exist? If so, what are its properties?
5. Does a mixed phase exist where the FP and the RP coexist?

Answering these questions is not only necessary for understanding the behavior in the asymmetric case, but also may bring some important insight into the SG problem.

It is our purpose to answer these questions by investigating the system size dependence of interfacial free energies calculated by a transfer matrix method.[4] The interfacial approach has been applied to the SG problem by several authors.[5,6,15,1] They obtained reasonable results confirmed later in spite of fairly small sizes used. This approach has also been developed in regular systems using Monte Carlo simulations so as to enable one to use finite-size scaling analyses.[15] It was shown to have various advantages compared with the conventional approach based on the calculation of linear responses such as susceptibilities (including correlation functions). In particular, it can investigate properties of ordered states as shown in both regular[16–18] and random[5,6,15,1] systems. We make use of the concentration and temperature dependences of two stiffness exponents relevant to the

FP and the RP which are defined for the size dependence of interfacial free energies averaged appropriately [see Eqs. (2.6) and (2.7)]. They represent measures of magnetic stiffness in the respective ordered states. Since they depend on p and T owing to critical fluctuations (which thus are effective values near criticality), one can extract some important information on fluctuations. Making use of it with the help of finite-size scaling,[19] we deduce the following interesting results in 2D which answer most of the questions made at the beginning in a unified way. The phase diagram of the RAS is determined clearly. Spatial fluctuations yield the vertical FP boundary, sharing the FP region with thermal fluctuations by the N-line. The RAS is a Mattis SG[20] with nonsingular continuation to the FP with respect to physical connectivity. We also consider percolative properties of the physical connectivity in ordered phases and the relations[8] between uniform magnetization and Edwards–Anderson order parameter.[1] They give further evidence for the result that the RAS in 2D is a Mattis SG which is obtained from stiffness exponents.

The paper is organized as follows. In Section 2, the stiffness exponents and amplitudes as well as interfacial free energies are defined and their general properties proper to random systems are given. Then we give a way to determine phase boundaries from the stiffness exponents and amplitudes. In Section 3 most of the numerical results in 2D and 3D are given and discussed, though they are only simply given for the 3D model. We also show a phase diagram obtained for the 2D model. In Section 4 we first give general properties of the stiffness exponent in the case of a pure ferromagnet, using finite-size scaling. Then, extending them to the random case, we derive properties of spatial and thermal fluctuations in the 2D FP and RAS from the numerical results given in Section 3, yielding a unified picture that describes the behavior of the FP and the RAS in 2D. In Section 5 we give further evidence for the RAS in 2D being a Mattis SG from the viewpoint of percolation and using Nishimori's relations. In Section 6 universal finite-size critical amplitudes of the interfacial free energy are examined and found to be consistent with the unified view obtained. In the last section we give some implications for properties of SG and RAS in $\geqslant$3D deduced from the results obtained in Sections 4 and 5 and give a summary and discussion.

2. STIFFNESS EXPONENTS AND PHASE BOUNDARIES

We consider the $\pm J$ Ising model in two and three dimensions with the Hamiltonian

$$H = -\sum_{\langle i,j \rangle} J_{ij} S_i S_j \qquad (S_i = \pm 1) \tag{2.1}$$

where the interactions $\{J_{ij}\}$ are a set of independent random values taking J and $-J$ only between nearest neighboring spins and have the distribution

$$P(J_{ij}) = p\delta(J_{ij} - J) + (1 - p)\,\delta(J_{ij} + J) \tag{2.2}$$

where p is the probability of the ferromagnetic coupling. Our systems are on square and simple cubic lattices whose sizes are $L+1$ (in units of the lattice constant) in the x direction and L in the other direction(s) (y when $d=2$, y and z when $d=3$). The spins at the boundaries in the x direction are fixed either all up (parallel boundary conditions, PBC) or up at one boundary and down at the other (antiparallel boundary conditions, APBC). In the remaining direction(s) periodic boundary conditions are imposed. Therefore, we have $L^{d-1}(L-1)$ spin variables in the d-dimensional lattice system. For a bond configuration $\{J\}$ in a system of linear size L, we denote the *total* free energy for the PBC as $F_L^{(\mathrm{p})}\{J\} = -T\ln\sum\exp(-H/T)$ (in units of $k_B = 1$) and that for the APBC as $F_L^{(\mathrm{ap})}\{J\}$. Then the interfacial free energy is defined as

$$\Delta F_L\{J\} = F_L^{(\mathrm{ap})}\{J\} - F_L^{(\mathrm{p})}\{J\} \tag{2.3}$$

Each bulk free energy is calculated by the numerical transfer matrix method.[4]

Although we call ΔF the interfacial free energy, it is not certain whether ΔF always reflects a single interface as in the pure Ising ferromagnets. If the external force exerted through BCs is conjugate to the order parameter of an ordered phase, then it makes an interface across the system. However, in random systems arbitrary BCs including the PBC and APBC are only incompletely conjugate to any order parameters. For the ordered phase without uniform magnetization, these BCs may rather make other distortions, such as fractional interfaces (or open domain walls) which are pinned by the boundaries. However, it still remains unchanged that ΔF is a response to the strong force applied through BCs[16]; it tells one whether or not and how the force is transferred from one to the other boundary, that is, whether a long-range order (LRO) exists or not and how stiff it is, by the property of how it goes to infinity or vanishes as $L \to \infty$. In general both the BCs yield some distortions or defects. If the PBC gives a smaller (larger) distortion than the APBC, then one gets $\Delta F_L\{J\} > 0$ (< 0) at low temperatures, hence a distribution of ΔF. Thus it is natural to consider its mean and mean deviation[15]:

$$W_L(p, T) = \langle \Delta F_L\{J\} \rangle_J \tag{2.4}$$

$$\tilde{W}_L(p, T) = [\langle \Delta F_L\{J\}^2 \rangle_J - \langle \Delta F_L\{J\} \rangle_J^2]^{1/2} \tag{2.5}$$

where $\langle \cdots \rangle_J$ denotes the bond average. We assume that only these two quantities are relevant to the phase transitions. $W_L(p, T)$ is relevant to the FP in such a way that $W_L(p, T)$ vanishes at the phase boundary. Since $\tilde{W}_L(p, T)$ arises owing to the randomness, we naturally assume that it is relevant to the RP. As it arises also in the FP, one cannot easily separate the contributions from the FP and the RP to see if they coexist or not. So one gets the phase boundary for the RP by $\tilde{W} \to 0$ only in the case where the FP is absent. We also assume that W and $\tilde{W}$ are only the singular parts in the critical region at least for a single ordered phase, which is justified later by their behavior above the critical temperature obtained in our calculations.

In order to treat these quantities in a qualitative way we introduce stiffness exponents, following Ueno *et al.*,[16]

$$W_L(p, T) = A(p, T)\, L^{a(p,T)} \tag{2.6}$$

$$\tilde{W}_L(p, T) = \tilde{A}(p, T)\, L^{\tilde{a}(p,T)} \tag{2.7}$$

We expect that W and $\tilde{W}$ vanish above the respective critical points for $L \to \infty$, or equivalently $a < 0$ and $\tilde{a} < 0$ for $L < \infty$. We also expect that a and $\tilde{a}$ become constant (a_0 and $\tilde{a}_0$) within the region of a single ordered phase relevant to it as $L \to \infty$; $a_0 = d - 1$ for the FP. In finite sizes the expectation holds asymptotically for L much larger than the correlation length (ξ, $\tilde{\xi}$ relevant to W, $\tilde{W}$, respectively). For $L < \xi$ and $L < \tilde{\xi}$, a and $\tilde{a}$ depend generally on p and T, but show little *explicit* dependence on L in the size extension of usual simulations as clarified in Section 4. We shall make use of these properties in Sections 4 and 5 to analyze the details of ordered phases.

In order to see the relation between the force and the interfacial free energy below the critical point, it is convenient to consider the total uniform magnetization at the boundaries $M_L^{(\alpha)}\{J\}$ ($\alpha = 1, 2$) that arises (at $T = 0$) when the PBC and APBC are replaced by the free BCs. Let the mean and mean deviation for them be BL^{ϕ} and $\tilde{B}L^{\tilde{\phi}}$, respectively. With the PBC and APBC imposed, then part of the force works on an LRO in proportion to $M^{(1)}$ or $M^{(2)}$, which yields $a = \phi$ and $\tilde{a} = \tilde{\phi}$. When an LRO is ferromagnetic, one obtains $\phi = d - 1$ and $\tilde{\phi} = \frac{1}{2}\phi$ in $L \to \infty$ (except at $p = 1$), because it is homogeneous and spatial fluctuations in the LRO become uncorrelated at large distances. When it is not ferromagnetic, $B = 0$ and $\tilde{B} \neq 0$, which gives only $\tilde{\phi} = \frac{1}{2}(d - 1)$ for $L \to \infty$. When the FP and RP coexist there are contributions to $\tilde{W}$ from both and usually one of them is dominant.

In finite systems one has to consider the finite-size effects which arise from inhomogeneity of an LRO because not only thermal fluctuations, but

also spatial fluctuations are not negligible. Leaving this problem in to Section 4, we consider qualitative relations between ϕ and $\tilde{\phi}$ for $L<\infty$ in the FP region. Let $l(p)$ be a correlation length that characterizes spatial fluctuations from the average absolute magnetization at $T=0$; $l(p)=\xi(p, T=0)$ given later in Section 4. Then $\tilde{\phi}=\frac{1}{2}\phi$ $(\phi=d-1)$ for $l(p)\ll L$. However, one may get $\tilde{\phi}>$ or $<\frac{1}{2}\phi$ for $l(p)\gg L$ even at $T=0$, thus yielding $\tilde{a}>$ or $<\frac{1}{2}a$, though both are effective values. Actually, $\tilde{a}>\frac{1}{2}a$ is found in the 2D FP near the phase boundary.

The following should be remarked. If only the FP exists, one obtains $a/\phi=\tilde{a}/\tilde{\phi}$ as the stiffness exponent for the *unit* strength of force; thus, $\tilde{a}=\frac{1}{2}a$ in $L\to\infty$. We have confirmed this result in the case of a 2D pure ferromagnet by applying random fields at the boundaries. Therefore $2\tilde{a}$ is the right expression of the stiffness exponent, which should be valid also for the RP. This is considered in Section 7 to examine previous studies on the SG.

According to the above discussions we obtain the following properties of stiffness exponents to determine the phase boundaries between the ordered phases and the disordered phase (DP):

$$a=0, \qquad \tilde{a}=0 \tag{2.8}$$

at the boundary between the FP and DP, and

$$a<0, \qquad \tilde{a}=0 \tag{2.9}$$

at the boundary between the RP and DP.

At the boundary of the FP related to the RP one may have

$$a=0, \qquad \tilde{a}>0 \tag{2.10}$$

which is in fact found in 2D and 3D. This suggests either of the following cases.

(i) The RP coexists with the FP if they are independent phases. This is the case as seen in the infinite-range model,[21] where there is a mixed phase of the SG and the FP.[22]

(ii) Both phases are only two different features of a kind of Mattis SG; in other words, they are not independent.

In the first case one is naturally interested in the phase boundary between both phases. One cannot use a and $\tilde{a}$ to find it, because a does not change and the FP dominates $\tilde{W}$ there, but the amplitude $A(p, T)$ of $W(p, T)$ is useful for it. When the two phases are independent they compete owing to the finite magnitude and the finite density of spin variables.

Then the appearance of the RP should reduce the amplitude unless the space occupied by the RP has dimensions less than $a+1$ in $L<\infty$. Thus, at the phase boundary between the competing RP and FP,

$$A(p, T) \text{ is singular} \tag{2.11}$$

in $L \to \infty$. For $L<\infty$ the singular behavior becomes diffuse, but we can still expect that the reduction gets more appreciable as one gets deeper inside the RP. This should be used in the region where A is determined without ambiguity.

3. NUMERICAL RESULTS

For the square lattice system of size $L=6$, 8, 10, 12, we calculated transfer matrices at almost 100 points in the p–T plane located at intervals of 0.01 in p and 0.15 in T. For the simple cubic system of size $L=3$, 4, the calculated points are almost 50 in number. For each given concentration we made calculations for 2000 (2D) and 4000 (3D) samples of bond configurations chosen at random and averaged over them to estimate $a(p, T)$ and $\tilde{a}(p, T)$ using least square fitting. We show the results in detail for 2D, but only roughly for 3D, since the sizes are too small. Results for $T=0$ in 2D are reported in a separate paper.[23]

3.1. Two Dimensions

Figure 1 shows the temperature dependence of $W_L(p, T)$ and $\tilde{W}_L(p, T)$ fixed with $L=10$ for $p \geqslant 0.86$. The W exhibits a similar monotonic decreasing function of T for all p given there. Thus, there is no crossing among the curves. On the other hand, as T is raised, $\tilde{W}$ follows a weakly decreasing curve for $0.94 \leqslant p \leqslant 0.98$ (of which only $p=0.98$ is given for clarity), while the decrease is larger and the slope changes only little for $p<0.94$ (only $p=0.86$, 0.88, 0.90 are given), thus showing crossings between both groups.

Figure 2a shows the T dependence of stiffness exponents a for $p \geqslant 0.86$. Every a is almost constant $a_0(p)$ at low temperatures a little below $T_m \sim 1.0$, where (p_m, T_m) is the multicritical point at which the FP and the RP phase boundaries meet, and it gradually decreases above T_m and, except for $p=0.86$ and 0.88, finally crosses the T axis. The T independence of a means inactive thermal fluctuations, as discussed in Section 4. From these results and other calculated ones we obtain the phase boundary for the FP given in Fig. 3, using Eqs. (2.8) and (2.10). The phase boundary is vertical except in the region near the multicritical point and the critical

concentration for the ferromagnetic order (at $T=0$) is $p_f=0.883$. The substitution of $p_f=0.883$ in the N-line equation $[\exp(-2J/T)=(1-p)/p]$ leads to the point $(p_\times=p_f,\ T_\times=1.01)$. These values deviate slightly from (p_m, T_m) which are given and discussed later. For $p=1$, $T_c=2.31$, which is about 2% larger than the rigorous value 2.269... .[24] We attribute this

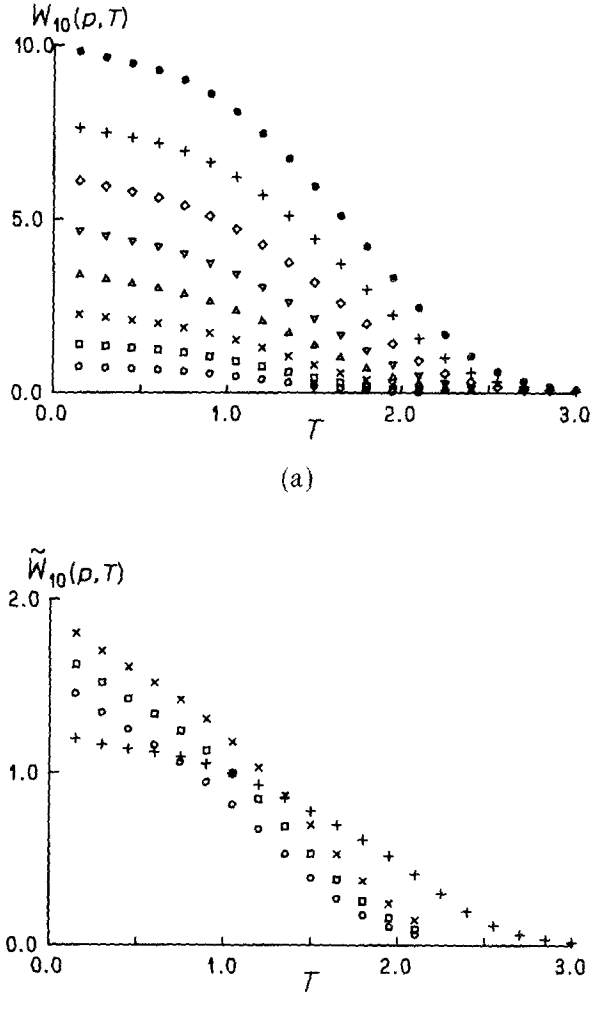

Fig. 1. (a) Mean $W_L(p, T)$ and (b) Mean deviation $\tilde{W}_L(p, T)$ of the interfacial free energy for $L=10$ vs. temperature T in 2D for concentrations $p=0.86$ (○), 0.88 (□), 0.90 (×), 0.92 (△), 0.94 (▽), 0.96 (◇), 0.98 (+), 1.0 (●). $\tilde{W}$ are given for some p's.

error to the smallness of the system size and the rigid boundaries. The latter always gives larger values and the error in T_c is proportional to $L^{-1/\nu}$ with a prefactor of $O(1)$.[19] Then it is reasonable to consider that the errors weakly depend only on p and T. In fact, the phase boundary given in Fig. 3 is very smooth, consistent with this.

In Fig. 2b each exponent $\tilde{a}$ for $p < 0.96$ also shows a T dependence similar to that for a, though it does not become constant at low tem-

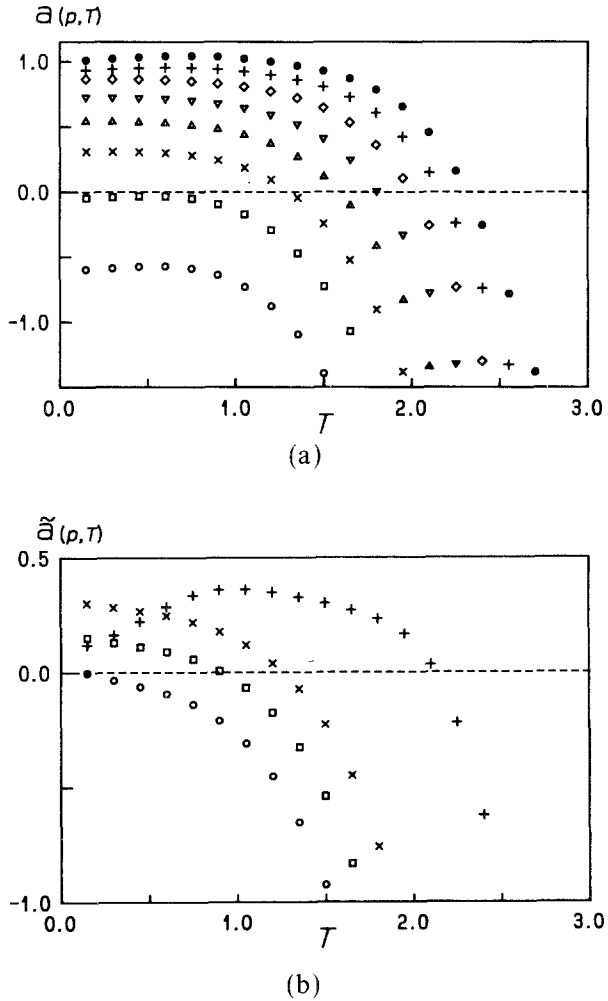

Fig. 2. (a) Stiffness exponents $a(p, T)$ and (b) $\tilde{a}(p, T)$ vs. T in 2D for $p = 0.86$ (○), 0.88 (□), 0.90 (×), 0.92 (△), 0.94 (▽), 0.96 (◇), 0.98 (+), 1.0 (●). $\tilde{a}$ are given for some p's.

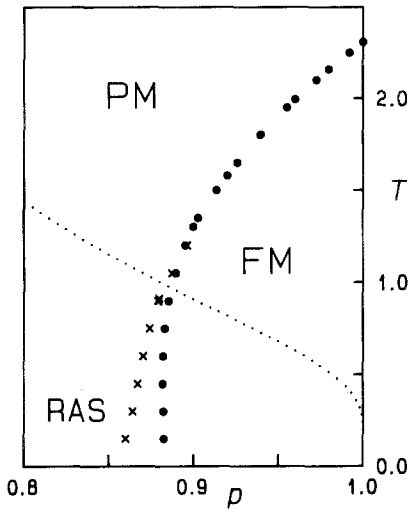

Fig. 3. Calculated phase diagram on the p–T plane for the two-dimensional model. Closed circles and crosses are obtained by Eqs. (2.8) and (2.9), respectively, and the dotted line is the Nishimori line.

peratures. This dependence is reasonable because usually the order becomes stiffer as T is reduced. On the other hand, for $p \geqslant 0.96$, $\tilde{a}$ exhibits unusual behavior in spite of the fact that most of the spins should be in the same direction. Since this behavior becomes more prominent as p gets close to 1, we cannot consider it intrinsic. It is probably because the average number of the antiferromagnetic bonds $2L(L-1)(1-p)$ is too small for small L and $1-p \ll 1$ to extract thermodynamic properties; it is 3.6 for $p=0.98$ and $L=10$. Therefore we will not discuss the cases of $p \geqslant 0.96$. It is also useful to see the p dependence of the exponents at $T=0.15$ in Fig. 4. One sees there apparently that Eq. (2.10) can be applied. The phase boundary of the RP for the DP is obtained using Eq. (2.9) as shown in Fig. 3. At $T=0.15$ it is present at $p \simeq 0.86$. According to work reported elsewhere,[23] $p_c(\mathrm{RP}) \simeq 0.854 \pm 0.002$ and $p_f = 0.885 \pm 0.001$ at $T=0$. Since our RP region at very low temperatures is very close to $0.85 \leqslant p \leqslant 0.9$ obtained by Barahona *et al.*[13] and Maynard and Rammal,[14] we assign the RP to the RAS. We note that $\tilde{a}$ increases with p across p_f to $p=0.94$, which is considered in Section 4 in studying the properties of the RAS.

The phase boundary of the RAS against the DP meets that of the FP at the multicritical point: $p_m \simeq 0.89$, $T_m = 1.05$. These are a little different from those at the point $(p_\times, T_\times)$. Since the errors become larger near the multicritical point because critical fluctuations in both phases take place, we can conclude that the phase boundary is vertical within the accuracy of

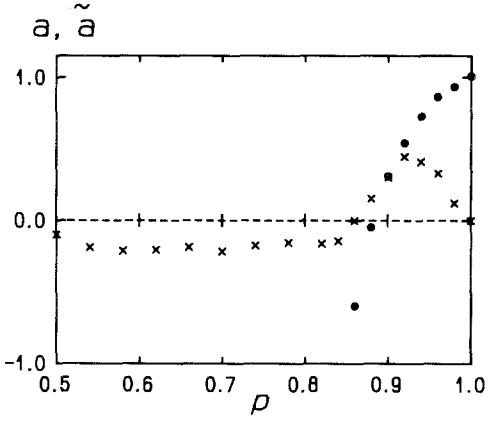

Fig. 4. Stiffness exponents a and $\tilde{a}$ vs. concentration p at $T=0.15$ in 2D with a (●) and $\tilde{a}$ (×).

the calculation. As to the RAS, although the errors of our calculations are not negligible, since they show the same tendency as already mentioned, we are assured of at least the existence of the single RP in a narrow region adjacent to the FP.

Figure 5 shows the amplitude $A(p, T)$ of W. For $p>0.88$, where the FP can exist, each A decreases gradually as T increases to the temperature where a is almost constant, $a_0(p)$, while it increases at higher temperatures. This property of weak T dependence below T_m is the consistent relation between A and a to be expected so long as the definition of A is valid. Each

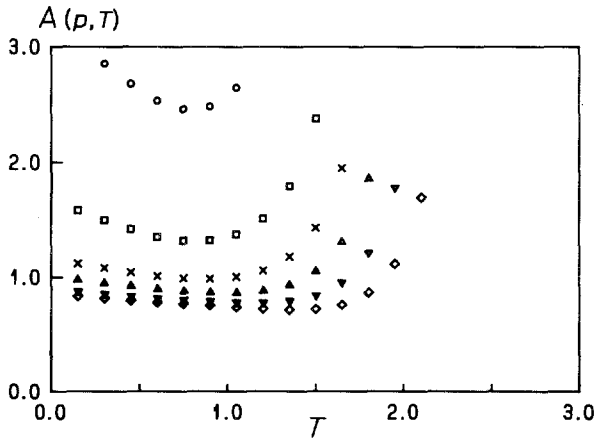

Fig. 5. Amplitude $A(p, T)$ of $W(p, T)$ vs. temperature for $p=0.86$ (○), 0.88 (□), 0.90 (×), 0.92 (△), 0.94 (▽), 0.96 (◇).

A does not exhibit any singular behavior and related reduction within the accuracy of calculation. From Eq. (2.11) this indicates that the RAS and FP are not competing.

Let us compare our results with those of others. All of our phase boundaries agree qualitatively with the one obtained by Ozeki and Nishimori.[25] Our result for the multicritical point is in fair agreement with theirs, $p_m \simeq 0.89$, $T_m \simeq 0.96$. Of interest is the critical concentration of non-frustrated plaquettes above which an infinite network of them develops; in other words, for $p < p_c(\text{nfp})$ there are fracture lines (or contours of zero energy) across the 2D system.[13,14] According to previous studies $p_c(\text{nfp}) = 0.86 \pm 0.02$,[26] 0.85.[13,14] These are close to our result $p = 0.86$ at $T = 0.15$ and also $p_c(\text{RP}) = 0.85$. This agreement is quite convincing because the interfacial free energy vanishes for $p < p_c(\text{nfp})$ in 2D unless entropy works for coupling as seen in highly degenerate regular models.[16,27] For the FP, p_f is also close to 0.88 ± 0.02,[4] 0.9,[28] 0.89 ± 0.02,[13] and 0.88.[12] Therefore the RAS exists at least between p_f and $p_c(\text{nfp})$. These properties on percolation are considered again in Section 5.

3.2. Three Dimensions

Figure 6 shows the T dependence of a and $\tilde{a}$. There are considerable errors in the results of a for $p = 1$: $T_c \simeq 5.1$ much larger than 4.511 obtained by high-temperature expansion[29] and $a > 2$ at $0.3 < T < 3.0$. Thus, our present results give only suggestions. We obtained $p_f \sim 0.75$ and $T_c \sim 1.5$, which are fairly close to the result $p_f = 0.767$ and $T_f = 1.86$ obtained by Monte Carlo simulations.[25] Figure 7 shows the p dependence of a and $\tilde{a}$ at $T = 0.3$. In the intermediate-p region $\tilde{a}(\sim 0.1) > 0$, which is consistent with the existence of the SG phase.[2,3,5-7] There is a narrow region which could be assigned to the RAS if the relative errors were almost the same for W and $\tilde{W}$ and weakly depend on p. Then, since Eq. (2.10) is satisfied and the T and p dependences of a and $\tilde{a}$ are similar to those in 2D, the same conclusion as in 2D is suggested: the RAS is a Mattis SG. It is interesting that $\tilde{a}$ seems continuous at the boundary of the RAS and SG.

Finally, we note that the decreasing T dependence of W and $\tilde{W}$ reveals little possibility of ordering due to entropy gains[16,27] in 2D and 3D, which has been considered for the mechanism of reentrant phase transitions by others. This is because, when the order of a low-temperature phase arises owing to entropy gains, ΔF has a positive slope at low temperatures, namely, $\Delta S = -\partial \Delta F / \partial T < 0$ as typically seen in 3D antiferromagnetic Potts models.[16] In addition, no coexistence of two ordered phases has been obtained in that case.[30]

4. STIFFNESS EXPONENTS AND FLUCTUATIONS

Following Fisher,[19] we assume finite-size scaling for the interfacial free energy,

$$W_L(p, T) = w(L/\xi(p, T)) \tag{4.1}$$

$$\tilde{W}_L(p, T) = \tilde{w}(L/\tilde{\xi}(p, T)) \tag{4.2}$$

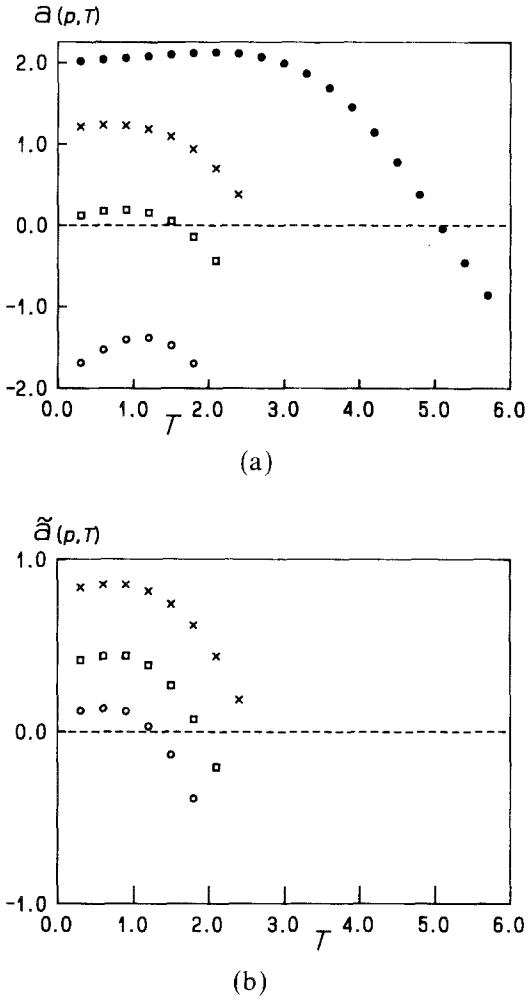

Fig. 6. Stiffness exponents (a) $a(p, T)$ and (b) $\tilde{a}(p, T)$ vs. temperature for the 3D model for concentrations $p = 1$ (●), 0.8 (×), 0.75 (□), 0.7 (○).

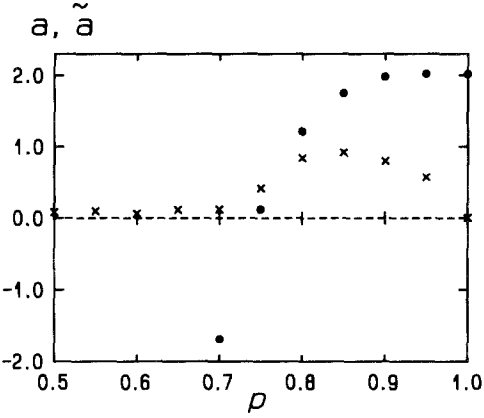

Fig. 7. Stiffness exponents a and $\tilde{a}$ vs. concentration at $T=0.3$ in 3D with a (●) and $\tilde{a}$ (×).

where ξ and $\tilde{\xi}$ are the correlation lengths characteristic for the FP and the RP, respectively. We also assume that Eq. (4.2) is valid whenever $\tilde{a} > \frac{1}{2}a$.

Before going to their properties, it is instructive first to discuss finite-size scaling in the pure ferromagnet,[16,19]

$$\Delta F_L(T) = f(tL^{1/\nu}) \tag{4.3}$$

where $t=(T-T_c)/T_c$ and $\xi \sim |t|^{-\nu}$, and the scaling function $f(x)$ has an asymptotic form

$$f(x) \sim \begin{cases} x^{a_0}, & x \to -\infty \\ f_0 + f_1 x, & x \to 0 \end{cases} \tag{4.4}$$

with $a_0 = d-1$. The condition $\xi \gg 1$ is necessary for Eq. (4.4) to hold. We also have another expression $\Delta F_L(T) = A(T)\, L^{a(T)}$, which is always valid with $a(T) = a_0$ for $\xi \ll L$ below T_c irrespective of whether $\xi \gg 1$ or not. Further, it is also approximately valid even for $\xi \gg L$ because the explicit dependence on L is usually negligible, that is,[18]

$$\left|\frac{\partial a}{\partial t}\right| \Big/ \frac{\partial a}{\partial L} = |t|/\nu L \ll 1$$

Thus, $a(T)$ is very useful from the following points of view. First, $a(T_c)=0$ gives T_c, consistent with finite-size scaling (4.3). Second, since $a=d-1$ for $L \gg \xi$ while $a<d-1$ and depends on T for $\xi > L$, then one easily obtains the crossover temperature where $\xi \simeq L$. Third, when $A(T)$ is little dependent on T, which appears a little below the crossover temperature, it means that $\xi \simeq O(1)$, namely, negligible thermal fluctuations. For the last

one can approximately use that $a(T)$ is independent of T instead of the property of A. We extend these analyses for the pure case to the random one, assuming the same asymptotic form as Eq. (4.4) for $w(x)$ and $\tilde{w}(x)$.

4.1. Fluctuations Related to $W(p, T)$ in 2D

Let us first focus on the region near the vertical boundary of the FP. We have obtained in Section 3 that $a(p, T)$ is independent of T and strongly dependent on p below about T_m and also $a < d-1$. It is reasonable to consider that $\xi(p, T)$ has almost the same properties as $a(p, T)$ except the last one. Then it follows that $\xi \gg L$ from its p dependence. Therefore spatial fluctuations are strong (which off course arise owing to frustrations), while thermal ones are negligible. These results imply the fractal nature of the FP in finite systems with fractal dimensions $d_f < d$, which is the result already obtained from a: $d_f = a + 1 < 0$.

The results obtained above on fluctuations in the FP lead to important properties of the FP as well as to reasonable interpretations of some properties previously obtained. The properties of fluctuations in the FP near the vertical boundary obtained above are nothing but a translation of the vertical boundary in terms of fluctuations or $\xi(p, T)$. This is because being vertical means absence of the effect of thermal fluctuations on the FP and thus a phase transition governed by spatial fluctuations; otherwise one should cross the boundary as T is raised with p fixed as seen above T_m. Thus, one obtains for the correlation length

$$\xi(p, T) \sim \varepsilon^{-\nu_p} \tag{4.5}$$

with critical exponent ν_p, where $\varepsilon = (p - p_m)/p_m$.

Thermal fluctuations are dominant in the thermal critical regime at high temperatures near the remaining boundary. Thus, the cross point and perhaps the N-line divide the two regimes. The results obtained above lead to the following expression of the finite-size scaling in each regime: $W_L(p, T) = w_T(x)$ with $x = tL^{1/\nu}$ for the thermal critical regime and $= w_p(y)$ with $y = \varepsilon L^{1/\nu_p}$ for the spatial critical regime. Here $w_T(x)$ and $w_p(y)$ have the same asymptotic form as Eq. (4.4). Near the multicritical point, $W_L(p, T)$ becomes Eq. (4.1) with $\xi(p, T)$ given by

$$\xi(p, T) = t^{-\nu} k(\varepsilon t^{-\phi}) \tag{4.6}$$

where t is replaced by $t = (T - T_m)/T_m$, $\phi = \nu/\nu_p$, and the scaling function $k(x)$ has an asymptotic form,

$$k(z) \sim \begin{cases} \text{const}, & z \to \infty \\ z^{-\nu_p}, & z \to 0 \end{cases} \tag{4.7}$$

There is a lowest value z_0 of z^- that satisfies $k(z_0) = \text{const}$. Then this $\varepsilon t^{-\phi} = z_0$ line should be equal to the N-line. Since the N-line is linear at the multicritical point irrespective of the dimensionality, then $\nu = \nu_p$. This is a required property because the FP is a single phase which includes no other phase with uniform magnetization, so that any critical behavior along its boundary should be the same except at the multicritical point.

We have obtained the interesting properties on the FP based on the results calculated in the 2D model, but we consider that those are unchanged in any higher dimensions because the vertical boundary and the N-line exist irrespective of the dimension.[8,9]

4.2. Fluctuations Related to $\tilde{W}(p, T)$ in 2D

We have obtained above that thermal fluctuations have little influence on the FP near the vertical boundary. Then $\tilde{a}$ also should be T independent, provided that the contribution to $\tilde{W}$ comes only from the FP. However, $\tilde{a}$ depends on T (see Fig. 1b), which reveals that $\tilde{\xi}(p, T) \gg L$ and the contribution comes only from the RP, i.e., the RAS. Thus, this indicates the existence of the RAS in the FP with $\tilde{\xi} \gg L$, which is consistent with the result (2.10) obtained in Section 3.

As already discussed in Section 2, there are two cases for this situation. There is a mixed phase in case (i) where both phases are independent. However, the absence of singularity in $A(p, T)$ obtained in Section 3 reveals a smooth variation of physical connectivity within the FP, rejecting case (i) and accepting case (ii). Further, case (ii) is also consistent with the p dependence of $\tilde{a}$: it increases with p near the vertical boundary where $\tilde{a} > \frac{1}{2}a$, while competeness should decrease $\tilde{a}$ as p increases. This case is also suggested by the harmonious relation between both phases seen in the phase diagram of Fig. 3. From this diagram one recognizes that the two boundaries with the DP obtained from Eqs. (2.8) and (2.9) form a naturally curved curve as a whole except around the multicritical point. Since the Mattis SG has no reference to spin directions, the average of the absolute interfacial free energy is relevant to it. Defining

$$W_L^{\mathrm{M}}(p, T) = \{W^2 + \tilde{W}^2\}^{1/2} \sim L^{a_{\mathrm{M}}(p, T)} \tag{4.8}$$

we also calculated the critical temperature from $a_{\mathrm{M}} = 0$. We obtained a smooth curve through the multicritical point, though the new boundary slightly extends the RAS region which is attributed to $W \gg$ or $> \tilde{W}$ even at points where $a < 0$.

5. FURTHER EVIDENCE FOR A MATTIS SPIN GLASS IN 2D

From the viewpoint of percolation let us show evidence for the argument that the RAS is a kind of Mattis SG. We have confirmed that the single RAS exists at $p_c(\text{nfp}) < p < p_f$ in 2D. So its order lies on an infinite network of nonfrustrated plaquettes,[13,14] that is, physical connection due to energy gains. Further, the RAS changes to the FP at p_f. Since the network gets larger and larger with p, this means the continuity of physical connectivity, in other words, the average value of absolute local magnetization is not zero at p_f and increases with no singularity with p. Thus, this is a percolation transition of the ferromagnetic component of the physical connectivity. This consideration is consistent with the property that the FP near its vertical boundary is governed only by spatial fluctuations obtained in Section 4.

We also have additional evidence for our argument. Suppose there is an infinite isotropic network of connected *sites* on a plane lattice and try to make another infinite network on the same lattice independent of the established one without disturbing it. Then it is impossible because, wherever one chooses the starting point for a network, it is completely surrounded by part of the established network; equivalently, one can always find two routes that connect two arbitrary sites (except those on the dangling part) on the established network and surround the starting point. Thus, it is only a kind of Mattis SG that is allowed to exist in the 2D system.

Furthermore, our argument is consistent with the following rigorous relations between m and Q obtained by Nishimori,[8] which hold inside the FP region in the thermodynamic limit:

$$m \begin{cases} > Q, & T > T_p \\ = Q, & T = T_p \\ < Q, & T < T_p \end{cases} \tag{5.1}$$

where $m = \langle\langle S_i \rangle_T \rangle_J$ (>0), $Q = \langle\langle S_i \rangle_T^2 \rangle_J$, and T_p is the temperature on the N-line with p. The Q is decomposed into a ferromagnetic part m^2 and a remaining part $Q - m^2$. The latter does not always represent the RP, but one can safely consider that the main contribution of $Q - m^2$ comes from the RP when $m^2 \ll Q - m^2$, because $Q - m^2$ does not change much as m approaches zero in the case where Q does not vanish simultaneously. Using Eq. (5.1), one obtains

$$Q - m^2 > m - m^2 \gg m^2 \tag{5.2}$$

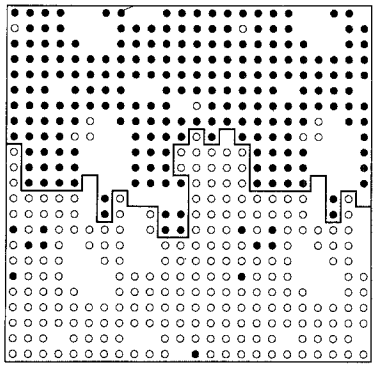

Fig. 8. A spin configuration of the ground state near the vertical FP boundary obtained previously[14]; closed and open circles represent up and down spins with rigid connectivity, and unmarked sites represent loose spins; the solid line is a magnetic wall which separates two domains. See ref. 14 for details.

in the FP near the vertical boundary where $m \ll 1$. Thus, the RP exists inside the FP region and is dominant over the FP near the boundary. Since the RP is the RAS in 2D, the above results are consistent with those obtained in Section 4.2 from the properties of $\tilde{a}$ and a. Figure 8 shows one of the ground states of the RAS obtained previously[13,14] near $p_f = 0.89$. It has large domains in up and down states. Thus, there is a large fluctuation $Q - m^2$ compared with the contribution m^2 from the mean near the vertical boundary.

6. EXAMINATION OF UNIVERSAL FINITE-SIZE CRITICAL AMPLITUDES

It is also of considerable interest to calculate the universal finite-size critical amplitude of the interface free energy at each phase boundary. It is considered to be a universal value independent of details within the same universality class.[31] Following Privman and Fisher, we define it as follows:

$$\begin{aligned} u_1(p_c, T_c) &= W_L(p_c, T_c)/T_c = \text{const} \qquad (T > T_m) \\ u_2(p_c, T_c) &= W_L(p_c, T_c)/T_m = \text{const} \qquad (T < T_m) \end{aligned} \tag{6.1a}$$

$$\tilde{u} = \tilde{W}_L(p_c, T_c)/p_c = \text{const} \tag{6.1b}$$

For u_2 we have taken into account that the relevant scaling field is only p, and u_1 and u_2 should agree at the multicritical point. For $\tilde{u}$, apparently p (pJ for the full notation) is an appropriate factor to guarantee inde-

pendence of concentration. Figure 9 shows the T dependences of these quantities. They are fairly large, but there are some characteristic features: as T increases, u and $\tilde{u}$ decrease smoothly except at and near T_m for u. This general tendency is probably due to the influence of the rigid boundaries on W and $\tilde{W}$ mainly through T_c. At $p=1$ we get $W(1, T_c=2.31)=2.88$ and this gives $u=0.62$, which is much smaller than the rigorous result: 1.530... for $T_c=2.269...$.[32] This suggests that the overestimation of T_c leads to a smaller value of u. We consider that the effect of rigid boundaries

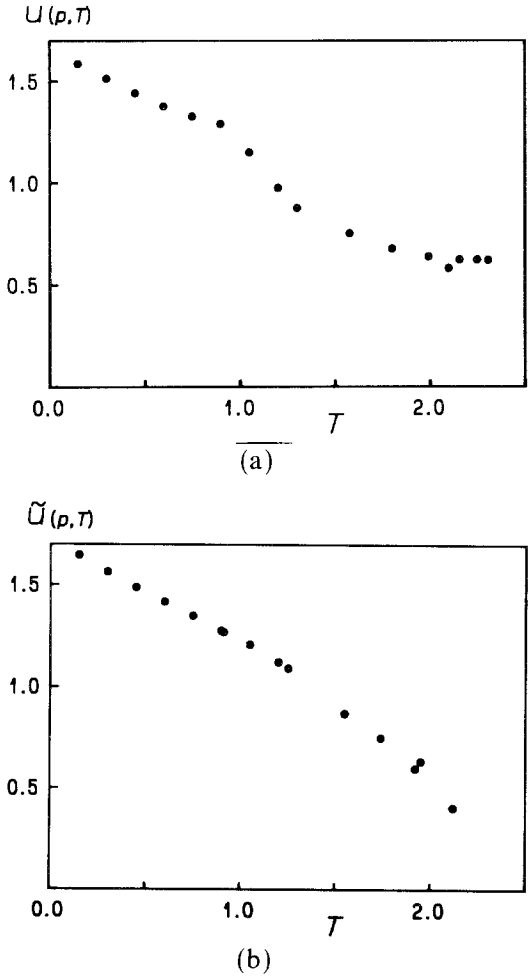

Fig. 9. Universal finite-size critical amplitudes of interfacial free energy. (a) u for mean W and (b) $\tilde{u}$ for mean deviation $\tilde{W}$ vs. temperature for the two-dimensional model; u is defined by dividing into u_1 for $T>T_m$ and u_2 for $T<T_m$ [see Eq. (6.1)].

depends on the degree of the connectivity at $T=0$, i.e., $a(p, 0)$ and $\tilde{a}(p, 0)$, which depend on p because of finite sizes. For larger $a(p, 0)$ the boundary force is transmitted with weaker decay even above the true T_c, pushing the critical point higher. Since for $L<\infty$, a and $\tilde{a}$ at very low temperatures decrease as p is reduced (recall that $\tilde{a}$ for $p>0.96$ has been excluded), then the rigid boundaries have a weaker effect on u and $\tilde{u}$ for smaller p.

Therefore these results are consistent with what we expect from the results obtained in Sections 4 and 5: u_1 and u_2 be constant and equal except at T_m and $\tilde{u}$ is constant including T_m. Since there occurs a crossover from the thermally-fluctuating regime to the spatially-fluctuating one at the multicritical point, u should be singular there. However, $\tilde{W}$ reflects the Mattis SG through T_m for $p<0.96$, so that $\tilde{u}$ should be nonsingular. Thus, the present results suggest the existence of universal finite-size critical amplitudes also in random systems.

7. COMMENT, SUMMARY, AND DISCUSSION

7.1. Comments on Properties of the RAS and the SG in $d>2$

If the vertical boundary of the FP exists in $d>2$ as argued by Nishimori,[9] then there must be no possibility except that this is a percolation transition only due to spatial fluctuations as obtained in 2D. It follows then that the RAS exists outside the vertical boundary as a Mattis SG. Our results in 2D and 3D suggest that the RAS region reduces as d increases and finally vanishes at $d=\infty$.[22]

We consider in general that there is no vertical phase boundary of an ordered phase whose order is associated only with physical connectivity due to energy gains. Otherwise, since the connectivity is critical at the boundary, there is a strong effect from thermal fluctuations and so the boundary cannot be vertical. This can also be applied to the SG. Thus the boundary of the SG cannot be vertical, so there are always reentrant spin glasses for all $d>2$ so long as the SG order arises owing to energy gains. It seems very unlikely that the SG order occurs owing to entropy gains at temperatures above the RAS boundary because then it must be disordered at lower temperatures below the RAS, which needs a high degree of ground-state degeneracy equivalent to a finite value of entropy.[16,26]

In Section 2, we obtained that $2\tilde{a}$ is the right stiffness exponent of the RP. In addition, we also obtained that spatial fluctuations are large even at very low temperatures around p_c(nfp) and p_f. Even in the intermediate-p region we consider that spatial fluctuations should be taken into account when one estimates the stiffness exponent from finite-size systems. Although

they are not always critical, they always exist in random systems. These inherent spatial fluctuations have a characteristic length $l(p=1/2)$ in the SG region discussed in Section 2. It is no doubt larger than the correlation length of frustrations which may be several lattice constants. Thus, if it is estimated in systems of $L < l(p)$, it is lower than the true value. Previously the stiffness exponent of the SG was calculated in small sizes to get $2\tilde{a}=0.17$ for $L=3\sim 6$[5] and $\tilde{a}=0.2$ for $L=2\sim 4$.[6] Thus, it is dangerous to use these values to discuss, say, the stability of the SG against an external field even if $2\tilde{a}$ is used.[33]

7.2. Summary and Discussions

Let us briefly summarize the present study to see how various properties are derived from the calculated results.

(i) $a=0$ and $\tilde{a}=0$ in 2D yield the phase boundaries between the FP, RAS, and the DP, which confirms the vertical FP boundary.

(ii) $a<d-1$ and its T independence in 2D yield dominance of spatial fluctuations in the ferromagnetic order near the vertical boundary, which is a close translation of the vertical boundary.

(iii) It follows then through scaling of ξ that the N-line divides the FP regime into two parts dominated by either of thermal and spatial fluctuations.

(iv) Together with the properties of stiffness amplitude A, the p and T dependences of a and $\tilde{a}$ reveal in 2D that the RAS is a Mattis SG and changes to the FP with nonsingular $\tilde{W}$ and singular W.

(v) The last result is further evidenced by percolation considerations and is consistent with the rigorous relations between m and Q.

(vi) The above results suggest even in $d>2$ that spatial fluctuations are dominant near the vertical FP boundary bounded by the N-line and that the RAS and the reentrant SG exist even in $d>2$.

(vii) It is pointed out that $2\tilde{a}$ is the right stiffness exponent for the RP and spatial fluctuations should be taken into account in random systems.

In the 2D case of the present study, we have obtained reasonable results for the sizes $L=6\sim 12$, though a much larger number of bond configurations are required to estimate reliable values of exponent ν. These results have revealed that the interfacial method is useful and powerful even in random systems. Thus, in order to obtain reasonable results in 3D one needs to calculate systems of sizes at least around $L=10$, including reliable data at $T \ll T_c$.

Finally we add the following comment.[34] Nishimori derived the vertical FP boundary by arguing that entropy for the distribution of frustrations on the N-line is singular at the multicritical point.[9] This argument is not inconsistent with holding that p_m is equal to the critical concentration p_f of the ferromagnetic order. There is the possibility of the existence of another singularity on the N-line corresponding to p_c(nfp).

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